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L_p-Equivalence of Impulsive Equations

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By means of a modification of Schauder's theorem, sufficient conditions for the L_p-equivalence of impulsive nonlinear differential equations are found.

1. INTRODUCTION

Sufficient conditions are found here for the integral equivalence (Marlin and Struble, 1969; Haščák and Švec, 1982; Simeonov and Bainov, 1984) of an impulsive nonlinear equation and one that is weakly perturbed in some sense with respect to the first one. The beginning of the qualitative investigation of impulsive equations was marked by the work of Mil'man and Myshkis (1960, 1963) and their theory in the finite-dimensional case is given in Samoilenko and Perestyuk, 1987).

2. STATEMENT OF THE PROBLEM

Let X be a Banach space and let $\mathbb{R}_+ = [0, \infty)$. By $\{t_n\}_{n=1}^{\infty}$ we denote a sequence of points $0 < t_1 < t_2 < \cdots$ satisfying the condition

$$\lim_{n \to \infty} t_n = \infty$$

Consider the impulsive equation

$$dx/dt = F(t, x) \qquad (t \neq t_n) \tag{1}$$

$$x(t_n + 0) = Q_n x(t_n) \qquad (n = 1, 2, 3, \ldots) \tag{2}$$

where $F(t, x)$: $\mathbb{R}_+ \times X \to X$ is a continuous function, $Q_n \in L(X)$ ($n = 1, 2, 3, \ldots$), and by $L(X)$ we have denoted the linear space of the linear bounded operators acting in X. Moreover, we assume that the operators

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Q_n are invertible, i.e., that there exist continuous inverse operators Q_n^{-1} $(n = 1, 2, 3, \dots)$.

Later, we consider the perturbed impulsive differential equation

$$dy/dt = F(t, y) + G(t, y) \qquad (t \neq t_n) \tag{3}$$

$$y(t_n + 0) = (Q_n + \Delta_n)y(t_n) \tag{4}$$

where $G(t, y)$: $\mathbb{R}_+ \times X \to X$ is a continuous function and $\Delta_n \in L(X)$ and $(Q_n + \Delta_n) \in L(X)$ are invertible operators.

Definition 1. We shall say that the function $\varphi(t)$ $(t \geq 0)$ is a solution of the equation (1)–(2) [(3)–(4)] if, for $t \neq t_n$, it satisfies equation (1) [(3)] and for $t = t_n$ the condition of "jump" (2) [(4)].

Let $1 \leq p \leq \infty$. By B_r we denote a closed ball in the space X with a center at zero and radius r.

Let $\mathbb{B}$ be an arbitrary Banach space and $\Omega \subset \mathbb{R}_+$. By $L_p(\Omega, \mathbb{B})$ we denote the space of functions x: $\Omega \to \mathbb{B}$ for which $\int_\Omega \|x(t)\|^p \, dt < \infty$. When $\mathbb{B} = \mathbb{R}$, we shall write $L_p(\Omega)$ and when $\mathbb{B} = \mathbb{R}$ and $\Omega = \mathbb{R}_+$, we shall write just L_p.

Definition 2. The equation (3)–(4) is called L_p-equivalent to the equation (1)–(2) in the ball B_r if there exists $\rho > 0$ such that for any solution $x(t)$ of (1)–(2) lying in B_r there exists a solution $y(t)$ of (3)–(4) lying in the ball $B_{r+\rho}$ and satisfying the relation $y(t) - x(t) \in L_p(\mathbb{R}_+, X)$. If equation (3)–(4) is L_p-equivalent to equation (1)–(2) in the ball B_r and vice versa, we shall say that equations (1)–(2) and (3)–(4) are L_p-equivalent in the ball B_r.

Definition 3. The equation (3)–(4) is called asymptotically L_p-equivalent to the equation (1)–(2) in the ball B_r if there exist numbers $\tau \geq 0$ and $\rho > 0$ such that for any solution $x(t)$ of (1)–(2) which is defined in $[\tau, \infty)$ and lies in the ball B_r there exists a solution $y(t)$ of (3)–(4) which is defined in $[\tau, \infty)$, lies in the ball $B_{r+\rho}$, and $y(t) - x(t) \in L_p([\tau, \infty), X)$. If equation (3)–(4) is asymptotically L_p-equivalent to equation (1)–(2) in the ball B_r and vice versa, then equations (1)–(2) and (3)–(4) are called asymptotically L_p-equivalent in the ball B_r.

3. MAIN RESULTS

3.1. Equivalent Equations

Set

$$q(t, \tau) = \prod_{t \leq t_j < \tau} Q_j^{-1} \tag{5}$$

Then each solution $x(t)$ of equation (1)-(2) which lies in the ball B_r is a solution of the nonlinear integral equation

$$x(t) = -\int_t^\infty q(t, s)F(s, x(s))\, ds \tag{6}$$

[provided that the right-hand side of (6) is defined].

Set

$$\tilde{q}(t, \tau) = \prod_{t \le t_j < \tau} (Q_j + \Delta_j)^{-1} \tag{7}$$

Then each solution $y(t)$ of equation (3)-(4) which lies in the ball B_ρ is a solution of the nonlinear integral equation

$$y(t) = -\int_t^\infty \tilde{q}(t, s)[F(s, y(s)) + G(s, y(s))]\, ds \tag{8}$$

[provided that the right-hand side of (8) is defined].

Set

$$z(t) = y(t) - x(t) \tag{9}$$

and subtract term by term equations (6) and (8).

Then the function $z(t)$ is a solution of the nonlinear integral equation

$$z(t) = -\int_t^\infty \{\tilde{q}(t, s)F(s, x(s) + z(s)) - q(t, s)F(s, x(s)) + \tilde{q}(t, s)G(s, x(s) + z(s))\}\, ds \tag{10}$$

or, more briefly,

$$z(t) = \Pi(x, z)(t) \tag{11}$$

where

$$\Pi(x, z)(t) = -\int_t^\infty \{\tilde{q}(t, s)F(s, x(s) + z(s)) - q(t, s)F(s, x(s)) + \tilde{q}(t, s)G(s, x(s) + z(s))\}\, ds \tag{12}$$

According to Definition 2, in order to establish the L_p-equivalence of equation (3)-(4) to equation (1)-(2) it suffices to show that for each solution $x(t)$ of equation (1)-(2) lying in the ball B_r the operator equation (11) has a fixed point $z(t)$ such that $x(t) + z(t) \in B_{r+\rho}$ for some $\rho > 0$ and which lies in $L_p(\mathbb{R}_+, X)$.

Hence the problem of finding sufficient conditions for the L_p-equivalence of equation (3)-(4) to equation (1)-(2) is reduced to the problem of the existence of a fixed point of the operator Π and the study of its properties.

In the proof of the existence of a fixed point of the operator Π in the present paper a modification of Schauder's classical principle is used referring to operators acting in the space $S(\mathbb{R}_+, X)$ of functions which are continuous for $t \neq t_n$ $(n = 1, 2, 3, \ldots)$, have at the points t_n limits from the left and the right, and are left continuous. The space $S(\mathbb{R}_+, X)$ is linear, locally convex, metrizable, and complete. A metric can be introduced to it, for instance, by means of the quality

$$\rho(x, y) = \|x - y\|$$

where

$$\|z\| = \sup_{0<T<\infty} (1+T)^{-1} \frac{\max_{0\leq t\leq T}\|z(t)\|}{1+\max_{0\leq t\leq T}\|z(t)\|} \tag{13}$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. It is not difficult to verify that for this space an analog of Ascoli–Arzella's theorem is valid: the set $M \subset S(\mathbb{R}_+, X)$ is relatively compact if and only if the intersections $M(t) = \{m(t)\colon m \in M\}$ are relatively compact for $t \in \mathbb{R}_+$ and M is equicontinuous on each interval $(t_{n-1}, t_n]$ $(n = 1, 2, 3, \ldots)$.

Lemma 1. Let the operator Π transform the set

$$C(r) = \{x \in S(\mathbb{R}_+, X)\colon x(t) \in B_r (t \in \mathbb{R}_+)\}$$

into itself and be continuous and compact.

Then Π has in $C(r)$ a fixed point.

3.2. Auxiliary Lemmas

In the further considerations we shall use some special properties of the linear integral operator

$$Qz(t) = -\int_t^\infty q(t, s) z(s)\, ds \tag{14}$$

in various spaces of functions which are defined on $\mathbb{R}_+$ and assume values in X. Since the integration everywhere is meant in the sense of Bochner, then the existence of the integrals follows from the respective estimates.

Lemma 2. Let $q(t, s)$ satisfy the inequality

$$\|q(t, s)\| \leq M e^{\delta(t-s)} \qquad (0 \leq t < s < \infty) \tag{15}$$

and $\delta > 0$.

Then $Q\colon L_p(\mathbb{R}_+, X) \to L_p(\mathbb{R}_+, X) \cap L_\infty(\mathbb{R}_+, X)$.

Proof. Since

$$\|Qz(t)\| \leq \int_t^\infty \|q(t, s)\| \cdot \|z(s)\|\, ds \leq M \int_t^\infty e^{\delta(t-s)} \|z(s)\|\, ds$$

then from Hölder's inequality it follows that

$$\|Qz(t)\| \leq M e^{\delta t}\left(\int_t^\infty e^{-\delta s p'}\, ds\right)^{1/p'} \|z\|_{L_p[0,\infty)}$$

$$= M e^{\delta t}\left(\frac{e^{-\delta t p'}}{\delta p'}\right)^{1/p'} \|z\|_{Lp[0,\infty)}$$

$$= M(\delta p')^{-1/p'}\|z\|_{L_p[0,\infty)} \qquad \left(\frac{1}{p}+\frac{1}{p'}=1\right)$$

Hence, for $z(t) \in L_p(\mathbb{R}_+, X)$ the function $Qz(t)$ is bounded and satisfies the inequality

$$\|Qz(t)\| \leq \frac{M}{(p'\delta)^{1/p'}}\|z\|_{L_p[0,\infty)} \tag{16}$$

We apply once more Hölder's inequality and obtain the estimate

$$\|Qz(t)\| \leq M\int_t^\infty e^{\delta(t-s)}\|z(s)\|\, ds$$

$$= M\int_t^\infty e^{\delta(t-s)/p'}\, e^{\delta(t-s)/p}\|z(s)\|\, ds$$

$$\leq M\left(\int_t^\infty e^{\delta(t-s)}\, ds\right)^{1/p'}\left[\int_t^\infty e^{\delta(t-s)}\|z(s)\|^p\, ds\right]^{1/p}$$

$$= M\delta^{-1/p'}\left[\int_t^\infty e^{\delta(t-s)}\|z(s)\|^p\, ds\right]^{1/p}$$

The above estimate implies the inequality

$$\left[\int_0^\infty \|Qz(t)\|^p\, ds\right]^{1/p} \leq M\delta^{-1/p'}\left\{\int_0^\infty \left[\int_t^\infty e^{\delta(t-s)}\|z(s)\|^p\, ds\right] dt\right\}^{1/p}$$

we apply Fubini's theorem and obtain

$$\left(\int_0^\infty \|Qz(t)\|^p\, ds\right)^{1/p} \leq M\delta^{-1}\left[\int_0^\infty \|z(s)\|^p\, ds\right]^{1/p}$$

Hence Q: $L_p(\mathbb{R}_+, X) \to L_p(\mathbb{R}_+, X)$.

Lemma 2 is proved. ■

Let $w(t)$ $(0 \leq t < \infty)$ be a scalar positive function which is integrable on each finite interval. Set

$$A(Q, w) = \left\{-\int_t^\infty q(t, s)z(s)\, ds\colon \|z(t)\| \leq w(t)\right\} \tag{17}$$

Lemma 3. Let inequality (15) hold and let, moreover, the function $q(t, s)$ be constant for $t \in (t_{n-1}, t_n]$ $(n = 1, 2, 3, \ldots)$.

Then the functions of $A(Q, w)$ are uniformly bounded and equicontinuous on $(t_{n-1}, t_n]$ for $n = 1, 2, 3, \ldots$.

Proof. The first assertion follows immediately from inequality (16). In order to prove the second assertion, we shall use the fact that for $t \in (t_{n-1}, t_n]$ the following equality holds:

$$-\int_t^\infty q(t, s)z(s)\, ds = -\int_t^\infty q(t_n, s)z(s)\, ds$$

For $t', t'' \in (t_{n-1}, t_n]$

$$\left[-\int_{t'}^\infty q(t', s)z(s)\, ds\right] - \left[-\int_{t''}^\infty q(t'', s)z(s)\, ds\right] = \int_{t''}^{t'} q(t_n, s)z(s)\, ds$$

which implies the estimate

$$\left\|\left[-\int_{t'}^\infty q(t', s)z(s)\, ds\right] - \left[-\int_{t''}^\infty q(t'', s)z(s)\, ds\right]\right\| \leq M \int_{t'}^{t''} e^{\delta(t_n - s)} w(s)\, ds$$

Since the function $w(s)$ is integrable on $(t_{n-1}, t_n]$, this implies the equicontinuity of the functions of $A(Q, w)$ on $(t_{n-1}, t_n]$.

Lemma 3 is proved. ■

From Lemma 3 and the above generalization of Ascoli-Arzella's theorem about the space $S(\mathbb{R}_+, X)$ there follows the compactness of the set $A(Q, w)$ when the space X is finite dimensional. In the general case when the space X is infinite dimensional, this is not true.

Let K be a convex, centrally symmetric, and closed set in X and let $w(t)$ be a scalar positive function defined on $\mathbb{R}_+$. Set

$$D(Q, w, K) = \left\{-\int_t^\infty q(t, s)z(s)\, ds \colon z \in L(w, K)\right\}$$

where

$$L(w, K) = \{z(t) \in S(\mathbb{R}_+, X) \colon \|z(t)\| \leq w(t),\ w^{-1}(t)z(t) \in K,\ t \in \mathbb{R}_+\}$$

Lemma 4. Let the following conditions be fulfilled:

1. The conditions of Lemma 3 hold.
2. The set $K \subset X$ is compact and centrally symmetric.
3. For $t \in \mathbb{R}_+$ the following inequality holds:

$$\int_t^\infty \|q(t, s)\| w(s)\, ds < \infty$$

Then the set $D(Q, w, K)$ is compact in $S(\mathbb{R}_+, X)$.

Proof. From Lemma 3 and the above generalization of Ascoli–Arzella's theorem about the space $S(\mathbb{R}_+, X)$ it follows that in order to prove Lemma 4, it suffices to verify that for $t \in \mathbb{R}_+$ the set of the intersections $D(Q, w, K)(t)$ of the functions of $D(Q, w, K)$ is compact.

Let $t \in \mathbb{R}_+$ be fixed. Then for any $T > t$ and $c > 0$ from the mean value theorem for integrals it follows that

$$\int_t^T q(t,s)z(s)[w(s)]^{-1}w_c(s)\,ds \in M(T-t)cK$$

where $w_c(s) = \min\{c, w(s)\}$. Since the set K is compact, then $M(T-t)cK$ is a compact subset of X. Moreover,

$$\left\| \int_t^T q(t,s)z(s)\,ds - \int_t^T q(t,s)z(s)w^{-1}(s)w_c(s)\,ds \right\| \le \int_{\Omega_c} \|q(t,s)\|w(s)\,ds$$

$[z \in L(w, K)]$, where $\Omega_c = \{s \in [t, T]: w(s) \ge c\}$. Since the function $\|q(t,s)\|w(s)$ is integrable on $[t, T]$, then

$$\int_{\Omega_c} \|q(t,s)\|w(s)\,ds \to 0 \qquad (c \to \infty)$$

This implies that the set

$$D_T = \left\{ \int_t^T q(t,s)z(s)\,ds \colon \|z(s)\| \le w(s),\ z(s)w^{-1}(s) \in K,\ s \in [t, T] \right\}$$

is approximated arbitrarily closely by the compact set $M(T-t)cK$, hence, by Hausdorff's theorem, is compact, too. Moreover, for $z \in L(w, K)$ the following inequality holds:

$$\left\| \int_t^\infty q(t,s)z(s)\,ds - \int_t^T q(t,s)z(s)\,ds \right\| \le \int_T^\infty \|q(t,s)\|w(s)\,ds$$

Since the function $\|q(t,s)\|w(s)$ is integrable on $\mathbb{R}_+$, then

$$\int_T^\infty \|q(t,s)\|w(s)\,ds \to 0 \qquad \text{for} \quad T \to \infty$$

But this means that the set $D(Q, w, K)$ is approximated arbitrarily closely by the compact sets D_T, and by Hausdorff's theorem it is compact itself.

Lemma 4 is proved. ∎

3.3. Conditions for L_p-equivalence

Before going on to the proof of the main assertions, we mention that all functions considered in the present paper are Bochner measurable. That is why a sufficient condition for their integrability is the absolute convergence of the respective integrals.

Theorem 1. Let the following conditions be fulfilled:

1. The operator-valued functions $q(t, s)$ and $\tilde{q}(t, s)$ satisfy the estimates

$$\|q(t, s)\|, \quad \|\tilde{q}(t, s)\| \leq M e^{\delta(t-s)} \qquad (0 \leq t < s < \infty) \tag{18}$$

where M, $\delta > 0$, and the operators Δ_n $(n = 1, 2, 3, \ldots)$ satisfy the condition

$$\sum_{t \leq t_k < s} \|\Delta_k\| \leq N e^{\varepsilon(t-s)} \qquad (0 \leq t < s < \infty) \tag{19}$$

where $N > 0$, $\varepsilon > -\delta$.

2. The function $F(t, x)$ satisfies the conditions

$$\varphi_r(t) = \sup_{\|u\| \leq r} \|F(t, u)\| \in L_p(\mathbb{R}_+) \tag{20}$$

$$\psi_{r,\rho}(t) = \sup_{\|u\| \leq r, \|v\| \leq \rho} \|F(t, u+v) - F(t, u)\| \in L_p(\mathbb{R}_+) \tag{21}$$

and

$$\psi_{r,\rho}^{-1}(t)[F(t, u+v) - F(t, u)] \in K \qquad \text{for} \quad t \in \mathbb{R}_+, \quad \|u\| \leq r, \quad \|v\| \leq \rho$$

where K is a convex, compact, centrally symmetric set and ρ is a positive number.

3. The function $G(t, x)$ satisfies the condition

$$\chi_{r+\rho}(t) = \sup_{\|u\| \leq r+\rho} \|G(t, u)\| \in L_p(\mathbb{R}_+) \tag{22}$$

and

$$\chi_{r+\rho}^{-1} G(t, u) \in K \qquad (0 \leq t < \infty, \|u\| \leq r)$$

4. The following inequality holds:

$$\frac{M^2 N}{[(\delta+\varepsilon)p']^{1/p'}} \|\varphi_r(t)\|_{L_p} + \frac{M}{(\delta p')^{1/p'}} \|(\psi_{r,\rho} + \chi_{r+\rho})(t)\|_{L_p} \leq \rho \tag{23}$$

Then the equation (3)-(4) is L_p-equivalent to the equation (1)-(2) in the ball B_r.

Proof. We shall show that for any function $x(t)$ such that $x(t) \in B_r$ $(t \in \mathbb{R}_+)$ the operator $\Pi(x, z)$ defined by equality (12) maps the set

$$C(\rho) = \{z \in S(\mathbb{R}_+, X) \colon z(t) \in B_\rho(t \in \mathbb{R}_+)\}$$

into itself.

Let $x(t) \in B_r$ $(t \in \mathbb{R}_+)$ and let $z \in C(\rho)$. Then the norm of the operator

$$\begin{aligned}\Pi(x, z)(t) = -\int_t^{\infty} \{&\tilde{q}(t, s)[F(s, x(s)+z(s)) - F(s, x(s))] \\ &+[\tilde{q}(t, s) - q(t, s)]F(s, x(s)) \\ &+\tilde{q}(t, s)G(s, x(s)+z(s))\}\, ds \end{aligned} \tag{24}$$

satisfies the estimate

$$\begin{aligned}\|\Pi(x, z)(t)\| \leq \int_t^{\infty} (&\|\tilde{q}(t, s)\| \cdot \|F(s, x(s)+z(s)) - F(s, x(s))\| \\ &+\|\tilde{q}(t, s) - q(t, s)\| \cdot \|F(s, x(s))\| \\ &+\|\tilde{q}(t, s)\| \cdot \|G(s, x(s)+z(s))\|\, ds\end{aligned}$$

whence, in view of (20) and (23), we obtain

$$\begin{aligned}\|\Pi(x, z)(t) \leq \int_t^{\infty} &\|\tilde{q}(t, s)\|[\psi_{r,\rho}(s) + \chi_{r+\rho}(s)]\, ds \\ &+\int_t^{\infty} \|\tilde{q}(t, s) - q(t, s)\|\varphi_r(s)\, ds\end{aligned} \tag{25}$$

The first of the integrals on the right-hand side of (25) can be estimated, in view of (18) and Lemma 2, in the following way:

$$\int_t^{\infty} \|\tilde{q}(t, s)\|[\psi_{r,\rho}(s) + \chi_{r+\rho}(s)]\, ds \leq \frac{M}{(\delta p')^{1/p'}} \|\psi_{r,\rho} + \chi_{r+\rho})(s)\|_{L_p} \tag{26}$$

In order to estimate the second integral in inequality (25), we use a suitable estimate of $\tilde{q}(t, s) - q(t, s)$. The expression $\tilde{q}(t, s) - q(t, s)$ satisfies the equality

$$\begin{aligned}\tilde{q}(t, s) - q(t, s) &= \prod_{t \leq t_j < s} (Q_j + \Delta_j)^{-1} - \prod_{t \leq t_j < s} Q_j^{-1} \\ &= -\sum_{t \leq t_k < s} \left[\prod_{t \leq t_j \leq t_k} (Q_j + \Delta_j)^{-1}\right] \Delta_k \left(\prod_{t_k \leq t_j < s} Q_j^{-1}\right) \\ &= -\sum_{t \leq t_k < s} \tilde{q}(t, t_{k+0})\Delta_k q(t_k, s)\end{aligned}$$

whence by (18) we obtain the estimate

$$\begin{aligned}\|\tilde{q}(t, s) - q(t, s)\| &\leq M^2 \sum_{t \leq t_k < s} e^{\delta(t - t_k) + \delta(t_k - s)} \|\Delta_k\| \\ &\leq M^2\, e^{\delta(t-s)} \sum_{t \leq t_k < s} \|\Delta_k\|\end{aligned}$$

In view of inequality (19) we obtain

$$\|\tilde{q}(t,s)-q(t,s)\| \le M^2 N e^{(\delta+\varepsilon)(t-s)}$$

Using the above inequality for the second integral in (25), we obtain the following estimate:

$$\int_t^\infty \|\tilde{q}(t,s)-q(t,s)\| \varphi_r(s)\,ds \le M^2 N \int_t^\infty e^{(\delta+\varepsilon)(t-s)}\,\varphi_r(s)\,ds$$

whence by Lemma 2

$$\int_t^\infty \|\tilde{q}(t,s)-q(t,s)\| \varphi_r(s)\,ds \le \frac{M^2 N}{[(\delta+\varepsilon)p']^{1/p'}} \|\varphi_r(s)\|_{L_p} \tag{27}$$

In view of inequalities (26), (27), and (25), for the expression $\|\Pi(x,z)(t)\|$ we obtain the estimate

$$\|\Pi(x,z)(t)\| \le \frac{M^2 N}{[(\delta+\varepsilon)p']^{1/p'}} \|\varphi_r\|_{L_p} + \frac{M}{(\delta p')^{1/p'}} \|\psi_{r,\rho}+\chi_{r+\rho}\|_{L_p}$$

whence by (23) we obtain $\|\Pi(x,z)(t)\| \le \rho$, i.e., $\Pi(x,z) \in C(\rho)$. Hence, for any $x \in C(r)$, the set $C(\rho)$ is invariant with respect to $\Pi(x,z)$.

We shall show that the operator $\Pi(x,z)$ is continuous in $S(\mathbb{R}_+, X)$.

First we establish that the set of values $\Pi(x,\cdot)$ on $C(\rho)$ is compact in $S(\mathbb{R}_+, X)$. In fact, the operators

$$F_x(z)(t) = F(t, x(t)+z(t)) - F(t, x(t))$$

and

$$G_x(z)(t) = G(t, x(t)+z(t))$$

transform $C(\rho)$, in view of (21), (22), into the sets $L(\psi_{r,\rho}, K)$ and $L(\chi_{r+\rho}, K)$, respectively, which, in view of Lemma 4, are further transformed by the linear integral operator $\tilde{Q}$ with a kernel $\tilde{q}(t,s)$ into compact sets.

Let the sequence $\{z_n(t)\} \subset C(\rho)$ be convergent in the metric of the space $S(\mathbb{R}_+, X)$ (i.e., uniformly on each bounded interval) to the function $z(t) \in C(\rho)$. Then, for $t \in \mathbb{R}_+$ the sequences $F(t, x(t)+z_n(t)) - F(t, x(t))$ and $G(t, x(t)+z_n(t))$ converge to $F(t, x(t)+z(t)) - F(t, x(t))$ and $G(t, x(t)+z(t))$, respectively. The two sequences of functions are majorized, respectively, by the functions $\psi_{r,\rho}(t)$, $\chi_{r+\rho}(t) \in L_p(\mathbb{R}_+)$. From Lemma 2 it follows that the convergent sequences of functions

$$\tilde{q}(t,s)[F(s, x(s)+z_n(s)) - F(s, x(s))], \qquad \tilde{q}(t,s)G(s, x(s)+z_n(s))$$

are majorized by the integrable functions $M e^{\delta(t-s)} \psi_{r,\rho}(s)$ and $M e^{\delta(t-s)} \chi_{r+\rho}(s)$, respectively. That is why within the integrals in formula (24) we may pass to the limit, hence $\Pi(x, z_n)(t)$ tends to $\Pi(x,z)(t)$ for

$t \in \mathbb{R}_+$. Since $\Pi(x, z)$ maps $C(\rho)$ into a compact set, this implies that $\Pi(x, z_n)$ tends to $\Pi(x, z)$ in $S(\mathbb{R}_+, Z)$ as well.

From Lemma 1 it follows that for any $x \in C(r)$ the operator $\Pi(x, z)$ has a fixed point z in $C(\rho)$, i.e., $z = \Pi(x, z)$. In view of Lemma 2, this fixed point belongs to the space $L_p(\mathbb{R}_+, X)$, i.e., equation (3)-(4) is L_p-equivalent to equation (1)-(2) in the ball B_r.

Theorem 1 is proved. ■

3.4. Conditions for Asymptotic L_p-Equivalence

Theorem 2. Let the following conditions be fulfilled.

1. The operator-valued functions $q(t, s)$ and $\tilde{q}(t, s)$ satisfy the conditions

$$\|q(t, s)\|, \quad \|\tilde{q}(t, s)\| \leq M e^{\delta(t-s)} \qquad (0 \leq t < s < \infty) \tag{28}$$

where M, $\delta > 0$ and the operators Δ_n $(n = 1, 2, 3, \ldots)$ satisfy the conditions

$$\sum_{t \leq t_k < s} \|\Delta_k\| \leq N e^{\varepsilon(t-s)} \qquad (0 \leq t < s < \infty)$$

where $N > 0$, $\varepsilon > -\delta$.

2. The function $F(t, x)$ satisfies the conditions

$$\begin{aligned} \varphi_r(t) &= \sup_{\|u\| \leq r} \|F(t, u)\| \in L_p(\mathbb{R}_+) \\ \psi_{r,\rho}(t) &= \sup_{\|u\| \leq r, \|v\| \leq \rho} \|F(t, u+v) - F(t, u)\| \in L_p(\mathbb{R}_+) \end{aligned} \tag{29}$$

where

$$\psi_{r,\rho}^{-1}(t)[F(t, u+v) - F(t, u)] \in K \qquad (t \in \mathbb{R}_+, \quad \|u\| \leq r, \quad \|v\| \leq \rho)$$

K is a convex, compact, centrally symmetric set and ρ is a positive number.

3. The function $G(t, x)$ satisfies the condition

$$\chi_{r+\rho}(t) = \sup_{\|u\| \leq r+\rho} \|G(t, u)\| \in L_p(\mathbb{R}_+) \tag{30}$$

and

$$\chi_{r+\rho}^{-1}(t) G(t, u) \in K \qquad (t \in \mathbb{R}_+, \quad \|u\| \leq r)$$

Then equation (3)-(4) is asymptotically L_p-equivalent to equation (1)-(2) in the ball B_r.

Theorem 2 is a consequence of Theorem 1. Its proof, by the substitution $\tilde{t} = t + \tau$, where τ is a sufficiently large, positive constant, is reduced to a verification of the conditions of Theorem 1.

Analogously, theorems on L_p-equivalence and asymptotic L_p-equivalence of equations (1)-(2) and (3)-(4) can be formulated and proved.

3.5. Remarks

Remark 1. Conditions 1–4 of Theorem 1 can be replaced by the following conditions:

$$\int_t^{\infty} \|\tilde{q}(t, s)\| [\|F(s, x(s)+z(s)) - F(s, x(s))\| + \|G(s, x(s)+z(s))\|]\, ds + \int_t^{\infty} \|q(t, s) - \tilde{q}(t, s)\| \cdot \|F(s, x(s))\|\, ds \leq \rho \tag{31}$$

for some $\rho > 0$;

$$\lim_{t''-t'\to 0} \sup_{t',t''\in(t_{n-1},t_n]} \sup_{\substack{\|z(s)\|\leq\rho \\ z(s)\in S(\mathbb{R}_+, X)}} \times \int_{t'}^{t''} \|\tilde{q}(t_n, s)\| [\|F(s, x(s)+z(s)) - F(s, x(s))\| + \|G(s, x(s)+z(s))\|]\, ds = 0 \tag{32}$$

for $x(t) \in S(\mathbb{R}_+, X)$, $\|x(t)\| \leq r$, $n = 1, 2, 3, \ldots,$ $z(t) \leq \rho$

Remark 2. Conditions 1–3 of Theorem 2 can be replaced by the following conditions:

$$\int_t^{\infty} \|\tilde{q}(t, s)\| [\|F(s, x(s)+z(s)) - F(s, x(s))\| + \|G(s, x(s)+z(s))\|]\, ds \leq \lambda(T, r, \rho) \tag{33}$$

$$x(t), z(t) \in S(\mathbb{R}_+, X), \qquad \|x(t)\| \leq r, \qquad \|z(t)\| \leq \rho, \qquad t \geq T$$

where

$$\lim_{T\to\infty} \lambda(T, r, \rho) = 0$$

$$\lim_{T\to\infty} \sup_{t\geq T} \int_t^{\infty} \|q(t, s) - \tilde{q}(t, s)\| \cdot \|F(s, x(s))\|\, ds = 0 \tag{34}$$

$$x(t) \in S(\mathbb{R}_+, X), \qquad \|x(t)\| \leq r, \qquad T \leq t < \infty)$$

$$\lim_{t''-t'\to 0} \sup_{t',t''\in(t_{n-1},t_n]} \sup_{\substack{\|z(s)\|\leq\rho \\ z(s)\in S(\mathbb{R}_+, X)}} \int_{t'}^{t''} \|\tilde{q}(t_n, s)\| [\|F(s, x(s)+z(s)) - F(s, x(s))\| + \|G(s, x(s)+z(s))\|]\, ds = 0 \tag{35}$$

$$x(t) \in S(\mathbb{R}_+, X), \qquad \|x(t)\| \leq r, \qquad n = 1, 2, 3, \ldots$$

Remark 3. Theorem 1 can be extended, without any changes in the proof, to impulsive functional differential equations of the type

$$dx/dt = Fx \qquad (t \neq t_n)$$
$$x(t_n+0) = Q_n x(t_n) \qquad (n = 1, 2, 3, \ldots)$$

and

$$dy/dt = Fy + Gy \qquad (t \neq t_n)$$
$$y(t_n+0) = (Q_n + \Delta_n) y(t_n) \qquad (n = 1, 2, 3, \ldots)$$

Here F and G are operators defined for all functions $x \in S(\mathbb{R}_+, X)$ such that $\|x(t)\| \leq r + \rho$ and $x(t) \in L_p(\mathbb{R}_+, X)$ for concrete $r, \rho > 0$ and $1 \leq p \leq \infty$.

In particular, the operator G can be chosen in the following three ways:

1. $Gx(t) = \int_0^t K(t, s, x(s))\, ds$
2. $Gx(t) = G_1(t, x(t), x(\Delta(t))$, where $\Delta(t)\colon \mathbb{R}_+ \to \mathbb{R}_+$
3. $Gx(t) = G_1(t, x(t), \max_{s \in E(t)} x(s))$

where $X = \mathbb{R}$ and $E(t)$ is a finite set depending on t "well enough."

Remark 4. An essential role in the proof of Theorem 1 (Theorem 2) is played by inequalities (18) [(28)]. Analogous theorems can be proved when in their place inequalities of the following form hold:

$$\|w(t, s)\|, \quad \|\tilde{w}(t, s)\| \leq M e^{\delta(t-s)} \qquad (0 \leq s \leq t < \infty)$$

where $M > 0$, $\delta < 0$, $w(t, s) = \Pi_{s \leq t_j < t} Q_j$, $\tilde{w}(t, s) = \Pi_{s \leq t_j < t} (Q_j + \Delta j)$.

Then, instead of integral equations (6) and (8), the following equations should be considered:

$$x(t) = q(t, 0)x(0) + \int_0^t q(t, s) F(s, x(s))\, ds$$

$$y(t) = \tilde{q}(t, 0)x(0) + \int_0^t \tilde{q}(t, s)[F(s, y(s)) + G(s, y(s))]\, ds$$

In this case the condition of the invertibility of the operators Q_n and $Q_n + \Delta_n$ is superfluous. One can consider a still more general situation when the operators Q_n and $Q_n + \Delta_n$ are such that the impulsive linear equations

$$dx/dt = 0 \qquad (t \neq t_n)$$
$$x(t_n+0) = Q_n x(t_n) \qquad (n = 1, 2, 3, \ldots)$$

and

$$dy/dt = 0 \qquad (t \neq t_n)$$
$$y(t_n+0) = (Q_n + \Delta_n) y(t_n) \qquad (n = 1, 2, 3, \ldots)$$

are exponentially dichotomous.

Remark 5. For $p = \infty$ instead of the space $L_\infty(\mathbb{R}_+, X)$ it is more convenient to consider its subspace $L^0_\infty(\mathbb{R}_+, X)$ consisting of functions tending to zero for $t \to \infty$. Equation (12) can be considered in this space if the functions $\varphi_r(t)$, $\psi_{r,\rho}(t)$, and $\chi_{r+\rho}(t)$ are elements of $L^0_\infty(\mathbb{R}_+, X)$. In this case the operator $\Pi(x, z)$ will be compact not only in $S(\mathbb{R}_+, X)$ but also in $L_\infty(\mathbb{R}_+, X)$. In this case for the proof of Theorem 1 one can use not only Schauder's theorem, but also the theory of the rotations of the continuous compact vector fields (e.g., the theorem of the odd vector fields (Krasnosel'skii and Zabreiko, 1984).

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